Singular perturbations and first order PDE on manifolds

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Abstract

In this note we present some results concerning the concentration of sequences of first eigenfunctions on the limit sets of a Morse-Smale dynamical system on a compact Riemanniann manifold. More precisely a renormalized sequence of eigenfunctions converges to a measure μ concentrated on the hyperbolic sets of the field. The set of all possible measure turns out to be a sum of a finite Dirac distributions localized at the critical point of the field and absolutely continuous measure with respect to the Lebesgue measure on each limit cycles : the coefficients which appear in the limit measure can be characterized using the concentration theory.

In the second part, certain aspects of some first order PDE on manifolds are studied. We study the limit of a sequence solutions of a second order PDE, when a parameter of viscosity tends to zero. Under some explicit assumptions on some vector fields, bounded and differentiable solutions are obtained. We exibit the role played by the limit sets of the dynamical systems and provide in some cases an explicit representation formula.

1 Introduction

(V_n, g) denotes a compact riemannian manifold of dimension $n \geq 2$, with no boundary and $\Delta_g = -\nabla_i \nabla^i$ is the Laplace-Beltrami operator. This note concerns the study of the operator $L_\epsilon = \epsilon\Delta + \sum_{i=1}^n b_i \partial_i + c$ acting on smooth functions, when the parameter ϵ converges to zero. b is a regular vector field and c is a positive function.

In the first part, we are interested to study the behavior of the first eigenfunction sequence u_ϵ associated to the the smallest eigenvalues $\lambda_\epsilon > 0$ of the operator $L_\epsilon(u_\epsilon) = \lambda_\epsilon u_\epsilon$, when the parameter ϵ converges to zero.

Some local results about these sequences are known on bounded domain of $\mathbb{R}^n$ (see Friedmann [3] [4] [5], Friedlin Ventcell' [2]) : when b has only one attracting point, u_ϵ converges uniformly on every compact set to a constant as ϵ converges to 0, but when the point is repulsive the sequence converges in the distribution sense to a Dirac distribution centered at this point.

The behavior of the first eigenvalue λ_ϵ as ϵ goes to zero is well known for a large class of dynamical system : λ_ϵ converges to some quantity called the topological pressure P. This number is characterized by a variational problem on the set of probability measures : when the field b has a finite number of hyperbolic invariant sets K, then the topological pressure is

$$P = \sup(h_\mu + \int_{V_n} (c - \frac{d \det D\phi_t^u}{dt}) d\mu \mid \text{ supp } \mu \subset K \text{ and } \mu\, \phi_t - \text{invariant}) \qquad (1)$$

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The set of measures considered here is all the measures with support in K and invariant by the flow. h_μ is metric entropy (see [9]), ϕ_t is the flow induced by the vector field b and $D\phi_t^u$ is the differential of ϕ restricted to the unstable bundle of K.

The topological pressure P is attained at a measure, called the equilibrium state (see [9] [1]). When the recurrence set of the field b consists of p hyperbolic sets K_i, $i = 1..p$ the measure μ is concentrated on the union of the $K_i's$. More precisely, $\mu = \sum_{i=1}^p p_i \mu_{K_i}$ where μ_{K_i} is the equilibrium state associate to K_i, $p_i \geq 0$ and $\sum_{i=1}^p p_i = 1$ (see theorem 3.4 in [10]).

Unfortunately this fruitful approach is not adapted to the study of the first eigenfunction problem since the equilibrium measures are not invariant by the flow, in general. However we shall see that the limit measures have their supports on the recurrent sets of the field b although they are not invariant by the flow.

In the second part, we gives some results about the behavior of the solutions of the equation $L_\epsilon u_\epsilon = f$ when ϵ goes to zero, for a given smooth positive function f. Our main interest is to find bounded C^1-solutions and to understand the interaction between the geometry of the characteristic curves (trajectories) of the field b and the behaviour of the solutions of $L_\epsilon u_\epsilon = f$ when ϵ goes to zero. The fascinating point here is that when ϵ goes to zero the elliptic equation $L_\epsilon u_\epsilon = f$ tends to a hyperbolic one. In this last case the singularities of the solutions propagate along the characteristic whereas, in the elliptic case there is no propagation of singularities.

The fields considered here are Morse-Smale (see [13] [14]), that is 1-) the recurent set consists of a finite number of hyperbolic stationnary points and periodic orbits 2-) the stable and unstable manifolds of the recurrent orbits are pairwise transversal (for all p,q, the unstable manifold $W^u(p)$ of p and the stable manifold $W^s(q)$ of q intersect transversally : $T_a(W^u(p)) \oplus T_a(W^s(q)) = T_a(V_n)$). A more general class of vector fields will be considered elsewhere.

Theorem 1 *Suppose that the first eigenvalue of the operator $\Delta_g + a$ is positive and a is a positive function with a finite number of minimum points which are not degenerate (in the sense of a Morse). Consider the first eigenvalue problem (which has the following variational formulation)*

$$\lambda_\epsilon = \inf_{u \in H_1(V_n)-\{0\}} \frac{\epsilon \int_{V_n} \mid \nabla u \mid^2 + au^2}{\int_{V_n} u^2} \tag{2}$$

Then, when ϵ converges to zero the sequence λ_ϵ converges to the minimum of the function a and the set of limits for the weak topology, when ϵ goes to zero, of the family of measures $\frac{u_\epsilon^2 dV_g}{\int_{V_n} u_\epsilon^2 dV_g}$ defined by the positive solutionu_ϵ of the PDE,

$$\epsilon \Delta_g u_\epsilon + a u_\epsilon = \lambda_\epsilon u_\epsilon \text{ on } V_n \tag{3}$$

is contained in the simplex $M = \{\nu = \sum_{i=1}^n c_i^2 \delta_{P_i}, \sum_{i=1}^n c_i^2 = 1\}$ of all probability measures with support in the finite set $\{P_i \mid i = 1..n\}$ where δ_P denotes the Dirac measure at the point P.

Remark : u_ϵ is uniquely defined up to a multiplicative constant by the Krein-Rutman theorem.

Consider now the case when $b = \nabla\phi$ and the function c is choosen so that the eigenvalue λ_ϵ of the operator $L_\epsilon = \epsilon\Delta + \sum_{i=1}^n b_i \partial_i + c$ is positive on the manifold. To study the family u_ϵ of solutions of the PDE

$$\epsilon \Delta u_\epsilon + \sum_{i=1}^n b_i \partial_i u_\epsilon + c u_\epsilon = \lambda_\epsilon u_\epsilon \text{ on } V_n, \tag{4}$$

we apply the transformation $b = \nabla\phi = -2\epsilon\nabla\psi_\epsilon$ and consider the new variable $v_\epsilon = u_\epsilon\psi_\epsilon$. Equation (4) is transformed into the following PDE where the vector field disappear:

$$\epsilon^2\Delta v_\epsilon + a_\epsilon v_\epsilon = \epsilon\lambda_\epsilon v_\epsilon \text{ on } V_n \tag{5}$$

where $a_\epsilon = c\epsilon + \frac{|\nabla\phi|^2}{4} + \frac{\epsilon\Delta\phi}{2}$.

Proposition 1 *Suppose that the following condition is satisfied : at each minimum points P_i of the function c, $c(P_i)+\Delta\phi(P_i)/2 \geq 0$. Let v_ϵ be a minimizer of the following variational problem*

$$\epsilon\lambda_\epsilon = \inf_{u\in H_1(V_n)-\{0\}} \frac{\epsilon^2\int_{V_n} \mid \nabla u \mid^2 + a_\epsilon u^2}{\int_{V_n} u^2} \tag{6}$$

then

- $\lim_{\epsilon\to 0}\lambda_\epsilon = \inf_{V_n}(\nabla\phi)^2 = 0$
- *The set of limits for the weak topology of the measures $v_\epsilon^2 dV_g$ is localized at the points where the function $\lim_{\epsilon\to 0}\min a_\epsilon$ is zero. These measures have support on the critical points of the vector field b.*

  Moreover $\sup_{V_n} v_\epsilon$ converges to $+\infty$

Going back to the sequence of eigenfunctions, we get that for some subsequence of ϵ tending to zero

$$\lim_{\epsilon\to 0}\frac{\int_{V_n} e^{-\phi/\epsilon}u_\epsilon^2 a_\epsilon}{\int_{V_n} e^{-\phi/\epsilon}u_\epsilon^2} = \min(\nabla\phi)^2 = 0 \tag{7}$$

and for any function $\psi \in C^\infty(V_n)$ we have for some subsequence of ϵ tending to zero

$$\frac{\int_{V_n} e^{-\phi/\epsilon}u_\epsilon^2\psi}{\int_{V_n} e^{-\phi/\epsilon}u_\epsilon^2} \to \sum_{i=1}^{n} c_i^2\psi(P_i) \tag{8}$$

where $c_i^2 = \lim_{\epsilon\to 0}\frac{\int_{B_{P_i}(\delta)} e^{-\phi/\epsilon}u_\epsilon^2}{\int_{V_n} e^{-\phi/\epsilon}u_\epsilon^2}$ (the limit is independant of δ). The set of weak limits of the family of measure $\frac{e^{-\phi/\epsilon}u_\epsilon^2}{\int_{V_n} e^{-\phi/\epsilon}u_\epsilon^2}dV_g$ is given by $\sum_{i=1}^n c_i^2\delta_{P_i}$ where P_i are the critical points of the function ϕ, the zero of the vector field $b = \nabla\phi$.

Next we will consider vector fields of the following form $b = -\nabla L + \Omega$ where Ω is fixed but not of gradient type and we will construct L to be a type of Lyapunov function of Ω. Using this vector field, we study the behavior of the sequence of the first eigenfunction for the operator L_ϵ and we obtain the following theorem :

Theorem 2 *On a compact Riemannian manifold V_n, consider a Morse-Smale vector field Ω not a gradient whose recurrent set consists of the stationnary points $P_1, ..., P_M$ and of the periodic orbits Γ_1,.., Γ_N. L denotes a special Lyapunov function associated with Ω defined in the proof of Theorem 2.*

For $\epsilon > 0$ let λ_ϵ, u_ϵ denotes respectively the first eigenvalue and an associated eigenfunction of the operator

$$\epsilon\Delta_g + b\nabla + c \text{ on } V_n \tag{9}$$

Then the set of all weak limits as ϵ goes to zero, of the family of normalized measures

$$\frac{e^{-L/\epsilon}u_\epsilon^2 dV_g}{\int_{V_n} e^{-L/\epsilon}u_\epsilon^2 dV_g} \tag{10}$$

is contained in the set $M = \{\nu \mid \nu$ *Borel measure, support* $\nu \subset \cup_{j=1}^N \Gamma_j \cup \{P_i \mid 1 \le i \le M\}$ *and* $\nu = \sum_{j=1}^N f_k^2(l_j)dl_j + \sum_{i=1}^M c_i^2 \delta_{P_i}\}$ *where* dl_j *denotes the arc length on* Γ_j*, for the metric* g*,* δ_{P_i} *the Dirac measure with support* P_i*,* $f_j : \Gamma_j \to \mathbb{R}$ *are continuous functions* $1 \le j \le N$ *and the* c_i *are constants.*

To prove the theorem, we will need 3 lemma

Lemma 1 *There exists a Lyapunov function L for the field Ω, such that L is twice differentiable in the neighborhood of the recurrent set of Ω and reaches its minimum on the union of the neighborhoods on the recurrent sets only. Outside these neighborhoods, L can be arbitrary such that* $\Psi(L) = \frac{1}{4}(\mid \nabla L \mid^2 + 2(\nabla L, \Omega)) \ge 0$

The same type of Lyapunov function L was constructed by Kamin [7] [8] in the case of an attractive point.

Lemma 2 *Under the assumptions of Theorem 2 on the vector field Ω,*

$$\lim_{\epsilon \to 0} \epsilon\lambda_\epsilon = 0 = \min_{V_n} \Psi \tag{11}$$

where $\Psi = \frac{(\nabla L)^2}{4} + \frac{(\Omega, \nabla L)}{2}$.

Lemma 3 *Under the assumptions of Theorem 2 on the vector field Ω, all weak limits of measures $v_\epsilon^2 dV_g$ as ϵ goes to zero are concentrated on the minimum set of the function Ψ.*

The measure μ is absolutely continuous with respect to the measure induced by the length along the periodic orbit. This is true on each pairwise orbit of the vector field. We have

$$\frac{d\mu}{dl} = f^2(l) = \lim_{\epsilon \to 0} \int_{H_l} u_\epsilon^2 d\Sigma_g \tag{12}$$

H_l denotes any hypersurface cutting the orbit at the point of abscisse l transversally. Remark: The limit is independant of the choice of the hypersurface.

2 First order PDE on manifolds

We study the limit of the solution u_ϵ of $L_\epsilon(u_\epsilon) = f$, where f is a given smooth function on the manifold, when ϵ tends to zero. The limit of the sequence when ϵ goes to zero, solves some first order PDE. For some previous works see [15], [16] and [11]. c and f will denote two given positive smooth functions and b a vector field. We suppose that $c_0 = \inf_{V_n} c > 0$ and $b_0 = \sup_{X \in TV_n} 1/2(\nabla_i b^k + \nabla_k b^i)(X_i, X_k)$ a finite number. S denotes the set of separatrices associates to the dynamical system. We prove the following theorem :

Theorem 3 *On a compact Riemannian manifold, consider a Morse-Smale vector field b and let c be a positive function satisfying $c(x) \ge c_0 > 0$ and $c_0 - b_0 > 0$. f is a differentiable*

function. Under these assumptions, there exists a solution $u \in C^0(V_n)$ such that $\mid \nabla u \mid \in L_\infty(V_n)$ of the first order PDE :

$$< b, \nabla u > + cu = f \text{ on } V_n \tag{13}$$

Moreover if the recurrent set of the vector field $b(x)$ consist of a finite number of points $P_1, ..P_p$, and $c = c_0$ is constant larger than the eigenvalues of Db, then u is in $C^1(V_n - S)$ and u is unique and is completely determined by the values $u(P_i) = \frac{f(P_i)}{c(P_i)}$. When b possess some limit cycles, the solution is not unique and has no limit near the limit cycles.

We adapt the previous theorem to the nonlinear first order PDE on a compact manifold.

$$< b(u,x), \nabla u > + c(u,x)u = f \tag{14}$$

where $b(\lambda, x)$ is a regular vector field, λ is parameter and $c(\lambda, x)$ and f are two given functions. The purpose of this part is to find some conditions on f,c and b to insure regular solutions, since there exists examples where shocks occur. To prove the existence of solutions for (14), we use an elliptic regularization and proceed by successive approximations, (see Jausselin et al. [12]).

For the last theorem we need the following notations :

$$b_0 = \frac{1}{2} \sup_{\|X\|=1, x \in V_n, \lambda \in \mathbb{R}} (\nabla_i b_k(x,\lambda) + \nabla_k b_i(x,\lambda))(X_i, X_k)$$

and $\gamma = \sup_{\lambda \in \mathbb{R}, x \in V_n} \mid \partial_\lambda b(\lambda, x) \mid$, $a_0 = \inf_{\lambda,x} c(\lambda,x) - b_0$. $A = \sup_{V_n} \mid \nabla f \mid + \sup_{V_n}(f/c) \times \sup_{V_n \times \mathbb{R}} \mid \frac{\partial c}{\partial x} \mid$ and $\beta = \sup_{V_n} \mid c' \mid . \sup_{V_n} \frac{f}{c}$, where we use the notation $c' = \frac{\partial c}{\partial \lambda}$.

We make the following assumptions for the rest of this section

1. $a_0 > \beta$ that is $\inf_{\lambda,x} c(\lambda,x) - b_0 > \sup_{V_n} \mid c' \mid \sup_{V_n} \frac{f}{c}$
2. $(\inf_{\lambda,x} c(\lambda,x) - b_0)^2 + (\sup_{V_n} \mid c' \mid \sup_{V_n} \frac{f}{c})^2 \geq 2(\inf_{\lambda,x} c(\lambda,x) - b_0)(\sup_{V_n} \mid c' \mid) \sup_{V_n} \frac{f}{c} + 4 \sup_{\lambda,x} \mid \partial_\lambda b(\lambda,x) \mid \times \sup_{V_n} \mid \nabla f \mid$
3. If $\Lambda = \inf_{\mathbb{R}\times\mathbb{R}}(c(\lambda,u) + uc'(\lambda,u) - b_0)$ and $\Lambda^2 - 4A\gamma \geq 0$

These assumptions mean that the minimum of the function c must be large enough with respect to the vector field b : it is a hyperbolicity condition.

Theorem 4 *On a compact manifold, consider the parametrized smooth vector field $b(\lambda, x)$, and let $c(\lambda, x)$ satisfie $c(\lambda,x) \geq c_0 > 0$ and c_0 large so that $c_0(c_0 - b_0) > \sup_{V_n}(f \mid c' \mid)$ and the conditions above satisfied. Then there exists a solution $u \in W^{\infty,1}(V_n)$ such that the first order PDE :*

$$< b(u,x), \nabla u > + c(u,x)u = f \text{ on } V_n \tag{15}$$

Moreover if the limit set of the vector field $b(u,x)$ is a union of a finite number of points $P_1, ..P_p$, u is in $C^1(V_n - S)$, where S is the set of separatrices of the field $b(u,x)$. If the number of solution of $c(u(P_i), P_i)u(P_i) = f(P_i)$ is finite and equal to k, the number of solutions u is k^p.

References

[1] M.D. Donsker S.Varadham On a varaitional formula for the principal eigenvalue for the operators with maximum principle, Proc. Nat. Acad. Sci. USA, 72, 1975,p780-783.

[2] M. Friedlin–A. Wentzell–Random Pertubations of Dynamical Systems, Springer-Verlag, 1984

[3] A. Devinatz–A. Friedman–The Asymptotic behavior of the solution of a singular perturbed Dirichlet problem *Indiana Univ. Math. J. 27, No. 3, 1978* p527-537.

[4] A. Friedman–The Asymptotic behavior of the First Real Eigenvalue of Second Order Elliptic Operators with a Small Parameter in the Highest Derivative, *Indiana Univ. Math. J. 22, No. 10, 1973* p1005-1015.

[5] A. Devinatz– R.Ellis–A. Friedman–The Asymptotic behavior of the First Real Eigenvalue of Second Order Elliptic Operators with a Small Parameter in the Highest Derivative, II *Indiana Univ. Math. J. 23, No. 11, 1974* p991-1011.

[6] A. Devinatz– A. Friedman– Asymptotic Behavior of the Principal Eigenfunction for a Singularly Perturbed Dirichlet Problem *Indiana Univ. Math. J. 27, No. 1, 1978* p143-157.

[7] S. Kamin–Exponential descent of solutions of elliptic singular pertubation problems, *Comm P.D.E, 9(2) 1984 p197-213.*

[8] S. Kamin– Singular pertubation problems and Hamilton-Jacobi *Integral Equations and Operator Theory, Vol. 9, 1986* p95-105.

[9] Y. Kifer– Principal eigenvalues, Topological pressure, and stochastic stability of equilibrium states, *Israel Journal of Math.Vol. 70, No.1 1990*

[10] Y. Kifer– Random Perturbations of Dynamical Systems, Birkäuser, 1988

[11] J. Moser– A rapidly convergent iteration method and non-linear partial differential equations -I *Ann. Scuola Norm. Sup. Pisa,8, 1965* p 290- 313.

[12] H.Jausselin–H.. Kreiss– J. Moser– On the forced Burgers equation with periodic Boundary conditions *Proceeding of the A.M.S. ,8, 1999* p 290- 313.

[13] C.Robinson– Dynamical Systems, CRC press *1995.*

[14] S. Smale–Differential Dynamical Systems, Bull. of A.M.S, 1967 p747-830.

[15] R. Sacker– A new approach to the pertubation theory of invariant surfaces *Comm. on Pure and Appl. math. VolXVIII, 1965 p.717-732* .

[16] R.Sacker–A pertubation theorem for invariant Riemannian manifolds, *Symp. on Diff Eq. Puerto Rico* 1965 p43-54.